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SB-3450

M. A. / M. Sc. (Part - I) Examination

March / April – 2011

Mathematics : Paper-402

(Functional Analysis)

(Old Course)

Time : 3 Hours] [Total Marks : 70

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :

☛ **M. A. / M. Sc. (Part - I)**

Name of the Subject :

☛ **Mathematics : Paper-402 (Old)**

☛ Subject Code No. : **3 4 5 0** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Answer all question.
(3) Each question carries 14 Marks.
(4) Follow usual notatins.

1 (a) Define a metric induced by norm on a vector space. Prove that every normed space is a metric space. **5**

(b) State and prove Riesz's lemma. **5**

(c) Prove that every finite dimensional sbuspace Y of a normed space X is complete. **4**

OR

1 (a) Define equivalent norms. Prove that the norms $\|x\|_\infty$ and $\|x\|_2$ on R^n defined respectively as $\|x\|_\infty = \max_j |\xi_j|$ and $\|x\|_2 = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2}$ for $x = (\xi_1, \xi_2 \ldots, \xi_n)$ are equivalent. **5**

(b) Let $T: X \to Y$ be a linear operator, where X, Y are vector spaces, then prove that range of T, R(T) is a vector space. **5**

(c) Prove that a compact subset of a metric space is closed and bounded. **4**

2 (a) Let $T: D(T) \to Y$ be a bounded linear operator with D(T) lies in X and X,Y are normed spaces. Prove that (i) $x_n \to x \Rightarrow Tx_n \to Tx$ (ii) N(T) is closed. **5**

(b) Prove that a linear functional $f: C[a,b] \to R$ defined by $f(x) = \int_a^b x(t)dt \; \forall x \in C[a,b]; \forall t \in [a,b]$ is bounded and $\|f\| = b - a$. **5**

(c) Prove that every linear operator on a finite dimensional normed space is bounded. **4**

OR

2 (a) Let X be a finite dimensional vector space and $E = \{e_1, e_2 \cdots, e_n\}$ be a basis of X. Then prove that the set $F = \{f_1, f_2 \cdots, f_n\}$ satisfying $f_i(e_j) = \delta_{ji}, (i, j =, 2, \ldots, n)$ is a basis for algebraic dual space X^* of X and $\dim X^* = \dim X$. **7**

(b) Prove that the dual space of R^n is R^n. **7**

3 (a) In an inner product space X, prove that **5**

(i) $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$

(ii) $x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(b) If Y is a closed subspace of a Hilbert space H, then prove that $H = Y \oplus Y^{\perp}$. **5**

(c) In an inner product space X, prove that $x \perp y \Leftrightarrow \|x + \alpha y\| = \|x - \alpha y\|$, for every scalar α. **4**

OR

3 (a) If Y is a closed subspace of Hilbert space H, then prove that $H = Y \oplus Y^{\perp}$. **5**

(b) Show that a Banach space $l^p (p \neq 2)$ is not a Hilbert space. **5**

(c) Define an orthogonal projection P on a Hilbert space H onto a closed subspace Y of H. Prove that P is linear and bounded. **4**

4 (a) Let $T : H_1 \to H_2$ and $S : H_1 \to H_2$, be a bounded linear operators, where H_1 and H_2 are Hilbert spaces. Then Prove that **5**

(i) $(S+T)^* \equiv S^* + T^*$

(ii) $(\alpha T)^* \equiv \overline{\alpha} T^*, \forall$ scalar α.

(b) State Parseval's relation in Hilbert space. If Parseval's relation holds in Hilbert Space H, then prove that M is total in H. **5**

(c) Orthonormalise an LI set $\{(1,1,1), (-1,0,1), (1,-1,0)\}$ by Gram-Schmidt process. **4**

OR

4 (a) Prove that every bounded linear functional f on a a Hilbert space H can be represented as $f(x) = \langle x, z \rangle$, where z is uniquely determined by f and has norm $\|z\| = \|f\|$. **5**

(b) Define an orthonormal set. Prove that an orthonormal set in Hilbert space is LI. **5**

(c) Define self adjoint operator. Let be a bounded linear operator on a Hilbert space H. If T is self adjoint, then prove that $\langle Tx, x \rangle$ is real $\forall x \in H$. **4**

5 (a) State and prove Banach fixed point theorem. 5

(b) Show that in a Banach fixed point theorem, the condition $d(Tx,Ty) \leq \alpha\, d(x,y)$ cannot be replaced by $d(Tx,Ty) < d(x,y);\ x \neq y$. 5

(c) Let $U: H \to H$ and $V: H \to H$ be two unitary operators on a Hilbert space H, then prove that UV is unitary and U is normal. 4

OR

5 (a) Let v be a continuous function on $J = [a,b]$ and $\kappa(t,\tau)$ be continuous on closed square $J \times J$ such that $|\kappa(t,\tau)| \leq c$. Let μ be a scalar such that $|\mu| \leq \frac{1}{c(b-a)}$. Then prove that the integral equation $x(t) = v(t) + \mu \int_a^b \kappa(t,\tau)\, x(\tau)\, d\tau$ has a unique solution x on J. 5

(b) State and prove fixed-point lemma. 5

(c) If S and T are normal operators satisfying $ST^* \equiv T^*S$ and $TS^* \equiv S^*T$, then $S+T$ is also normal. 4